Lagrangian Averaged (LA) Stochastic advection by Lie transport (LA SALT)

Darryl D. Holm

Department of Mathematics
Imperial College London

CliMathParis IHP 2019

Climate – what is it?

"Climate is what you expect. Weather is what you get." [1]

> *There are many questions regarding climate whose answers remain elusive.*
>
> *For example, there is the question of determinism; was it somehow inevitable at some earlier time that the climate now would be as it actually is?*

An *almost intransitive system* is one that can undergo two or more distinct types of behaviour, and will exhibit one type for a long time, but not forever. What can we say about stochastic "almost intransitive systems"?

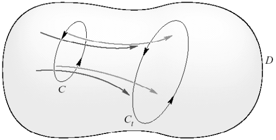
[1] Lorenz, E. N., 1995: Climate is what you expect. Unpublished, available at `http://eaps4.mit.edu/research/Lorenz/Climate_expect.pdf`

Lorenz, E. N., 1976: Nondeterministic theories of climatic change. Quaternary Research, 6(4), 495-506.

Daron, J.D. and Stainforth, D.A., 2013. On predicting climate under climate change. Environmental Research Letters, 8(3), p.034021.

Imperial College London

Right ... OK, what is Kelvin's theorem for the climate?



Q1: What do you mean by circulation?

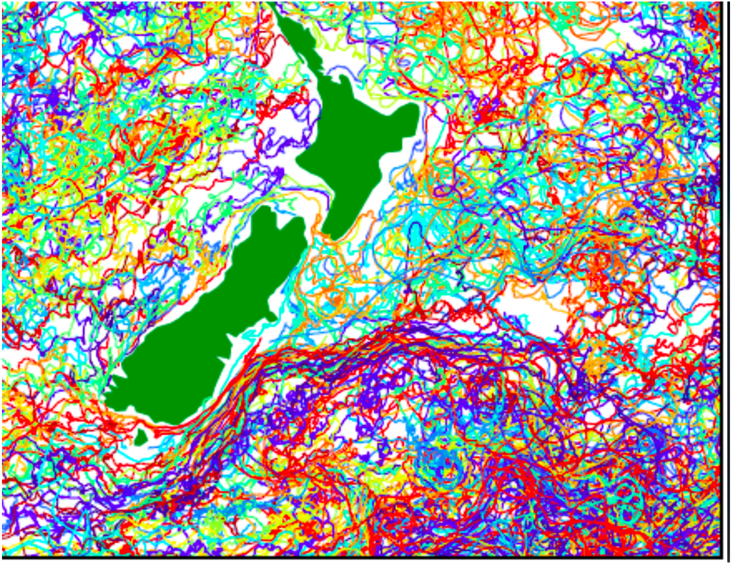
A1: As usual, circulation means, "integral of the momentum per unit mass (a 1-form) around a closed loop moving with the fluid velocity".

A2: Ah! Circulation would still be defined by the same formula, but now the loop would be moving with the fluid along the expected path in an ensemble of stochastic Lagrangian paths responding to Newton's Law?

Q3: Ah! So the expectation of the drift velocity of the stochastic ensemble of path velocities would be taken at fixed Lagrangian label on the loop?

Q4: And the expected loop would stay together even as an ensemble of stochastic paths with a shared expected drift velocity because the flow map preserves neighbours? Would that work?

Intuition solves problems by envisioning the solution. What would a stochastic Lagrangian trajectory look like?



Imperial College London

A Stochastic Kelvin Circulation Theorem with a Lagrangian Averaged (LA) Drift Velocity [DHL19]

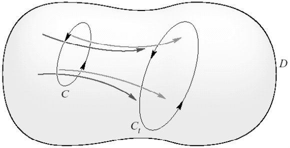
Suppose the divergence-free advection velocity is the stochastic process with a Lagrangian Averaged drift velocity?

$$\widetilde{u} := \underbrace{\mathbb{E}\,[u]\,(x,t)\,dt}_{\text{EXPECTED DRIFT}} + \sum_k \underbrace{\xi_k(x) \circ dW_k(t)}_{\text{NOISE}}, \quad \operatorname{div} \widetilde{\mathbf{u}} = 0.$$

Let $\mathbf{v}$ = momentum/mass. (In Hamilton's principle, $\mathbf{v} = D^{-1}\delta\ell/\delta\mathbf{u}$.)

The **stochastic Kelvin circulation theorem** represents **Newton's law** for the evolution of momentum concentrated on an advecting loop

$$\mathbf{d}\oint_{c(\widetilde{u})} \mathbf{v}\cdot d\mathbf{x} = \oint_{c(\widetilde{u})} \underbrace{(\mathbf{d} + \mathcal{L}_{\widetilde{u}})(\mathbf{v}\cdot d\mathbf{x})}_{\text{By KIW formula}} = \oint_{c(\widetilde{u})} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$



Imperial College London

The Lagrangian Averaged (LA) SALT equations

The SALT equations introduced earlier read, with $(\frac{\delta \ell}{\delta u} \in \mathfrak{X}^*,\ \frac{\delta \ell}{\delta a} \in V)$,

$$\mathbf{d}\frac{\delta \ell}{\delta u} + \mathcal{L}_{\mathbf{d}x_t}\frac{\delta \ell}{\delta u} \overset{\mathfrak{X}^*}{=} \frac{\delta \ell}{\delta a} \diamond a\, \mathbf{d}t \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}x_t} a \overset{V^*}{=} 0,$$

where $\mathbf{d}x_t := u_t(x_t)\, \mathrm{d}t + \xi(x_t) \circ \mathrm{d}W_t$ is a stochastic transport vector field. Now, let's replace the Eulerian drift velocity $u_t(x)$ in the stochastic transport vector field $\mathbf{d}x_t$ by its expectation, denoted as $\mathbb{E}\left[u_t\right](x)$, so that

$$\mathbf{d}X_t := \mathbb{E}\left[u_t\right](x)dt + \sum_k \xi^{(k)}(x) \circ dW_t^{(k)}$$

and let's consider the following modified Euler–Poincaré equations

$$\mathbf{d}\frac{\delta \ell}{\delta u} + \mathcal{L}_{\mathbf{d}X_t}\frac{\delta \ell}{\delta u} \overset{\mathfrak{X}^*}{=} \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\ \mathrm{d}t \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}X_t} a \overset{V^*}{=} 0\,.$$

The above equations comprise the class of LA SALT theories.

Imperial College London

Hamiltonian LA SALT, same Lie–Poisson matrix operator

We pass from the Lie–Poisson form of the SALT equations to the corresponding LA SALT form by setting

$$\frac{\delta(\mathbf{d}h)}{\delta\mu} = \mathbf{d}X_t = \mathbb{E}\left[\frac{\delta h}{\delta\mu}\right]\mathrm{d}t + \sum_k \xi^{(k)} \circ \mathrm{d}W_t^{(k)} \quad \text{and} \quad \mathbb{E}\left[\frac{\delta H}{\delta a}\right] = -\,\mathbb{E}\left[\frac{\delta \ell}{\delta a}\right].$$

Taking these expectations transforms the LA SALT equations from their Euler–Poincaré form above into their Hamiltonian form with the *same* Lie–Poisson matrix operator, as

$$\mathbf{d}\begin{bmatrix}\mu \\ a\end{bmatrix} = -\begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a \\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\mathbb{E}\left[\delta h/\delta\mu\right]dt + \sum_k \xi^{(k)} \circ \mathrm{d}W_t^{(k)} \\ \mathbb{E}\left[\delta h/\delta a\right]dt\end{bmatrix}.$$

Since the Lie–Poisson matrix operators for DALT, SALT and LA SALT are the all the *same*, they share the same Casimirs and Lagrangian invariants!

Imperial College London

The LA SALT expectation dynamics separates & closes

Upon converting LA SALT from Stratonovich into Itô form, we find

$$\mathbf{d}\mu + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}\mu dt + \mathcal{L}_{\xi^{(k)}}\mu dW_t^{(k)} = \left(\frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu)dt - \mathbb{E}\left[\frac{\delta H}{\delta a}\right]\diamond a\right)dt\,,$$

$$\mathbf{d}a + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}a dt + \mathcal{L}_{\xi^{(k)}}a dW_t^{(k)} = \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a)dt\,.$$

Applying the expectation to these equations yields the *PDE sub-system*,

$$\partial_t \mathbb{E}\left[\mu\right] + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}\mathbb{E}\left[\mu\right] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}\left[\mu\right]) = -\,\mathbb{E}\left[\frac{\delta H}{\delta a}\right]\diamond \mathbb{E}\left[a\right]\,,$$

$$\partial_t \mathbb{E}\left[a\right] + \mathcal{L}_{\mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]}\mathbb{E}\left[a\right] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}\left[a\right]) = 0\,.$$

This sub-system of PDEs for the expectation variables will be *closed* in certain cases of physical interest, some of which we will discuss later.

Interim summary

The nonlocality in probability space (à la McKean) in the LA SALT equations simplifies the dynamics of SALT in three significant ways.

(1) The Casimirs are still preserved by the full LA SALT dynamics, while the equations for the expected physical variables separate into a *dissipative sub-system* embedded into the larger conservative system.

(2) In many cases (including for the LA SALT incompressible Euler fluid) the fluctuation equations are *linear stochastic equations* whose solutions are transported and accelerated by forces involving the expected variables.

(3) In some cases, such as the 2D LA SALT Euler–Boussinesq (EB) equations, this linear stochastic transport property implies *unique global existence*, which is not possessed by the EB SALT equations.

Fluctuation dynamics

The fluctuation variables are defined as

$$\mu' := \mu - \mathbb{E}[\mu] \in \mathfrak{X}^*, \quad a' := a - \mathbb{E}[a] \in V.$$

Taking the difference between the Itô forms and the expectation equations yields the Itô fluctuation equations

$$\mathrm{d}\mu' + \mathcal{L}_{\mathbb{E}\left[\frac{\delta h}{\delta \mu}\right]}\mu' dt + \mathcal{L}_{\xi^{(k)}}\mu\, dW_t^{(k)} = \left(\frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu') - \mathbb{E}\left[\frac{\delta h}{\delta a}\right]\diamond a'\right)dt,$$

$$\mathrm{d}a' + \mathcal{L}_{\mathbb{E}\left[\frac{\delta h}{\delta \mu}\right]}a' dt + \mathcal{L}_{\xi^{(k)}}a\, dW_t^{(k)} = \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a')dt\,.$$

We pair these two equations with their corresponding dual variables to obtain stochastic evolution equations for the resulting quadratic quantities.

Imperial College London

Fluctuation variance dynamics

We then take expectation and integrate in space to find variance equations

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|\mu'|^2_{\mathfrak{X}}\right] &- \left\langle \mathbb{E}\left[\mathcal{L}_{\mu'\sharp}\mu'\right], \mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]\right\rangle_{\mathfrak{X}} + \left\langle \mathbb{E}\left[\mathcal{L}_{\mu'\sharp}a'\right], \mathbb{E}\left[\frac{\delta H}{\delta a}\right]\right\rangle_{\mathfrak{X}} \\
&= -\frac{1}{2}\sum_k \left\langle \mathbb{E}\left[\mathcal{L}_{\mu'\sharp}(\mathcal{L}_{\xi^{(k)}}\mu') + \mathcal{L}_{\left(\mathcal{L}_{\xi^{(k)}}\mu\right)^\sharp}\mu\right], \xi^{(k)}\right\rangle_{\mathfrak{X}}, \\
\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|a'|^2_{V}\right] &- \left\langle \mathbb{E}\left[\widehat{a'}\diamond a\right], \mathbb{E}\left[\frac{\delta H}{\delta \mu}\right]\right\rangle_{\mathfrak{X}} \\
&= -\frac{1}{2}\sum_k \left\langle \mathbb{E}\left[\widehat{a}'\diamond(\mathcal{L}_{\xi^{(k)}}a') + \widehat{\mathcal{L}_{\xi^{(k)}}a}\diamond a\right], \xi^{(k)}\right\rangle_{\mathfrak{X}},
\end{aligned}
$$

where $\mu'^\sharp \in \mathfrak{X}$ is dual to $\mu' \in \mathfrak{X}^*$ and $\widehat{a'} \in V^*$ is dual to $a \in V$.

- The *dynamics of the variances* of the stochastic system is driven by an intricate variety of correlations among the evolving fluctuation variables.
- The solution behaviour can be seen more easily in examples.

Analytical results for LA SALT Euler

The LA SALT Euler equation is given as

$$\mathbf{d}u_t + \mathcal{L}^T_{\mathbb{E}[u_t]} u_t dt + \sum_k \mathcal{L}^T_{\xi^{(k)}} u_t \circ dW_t^{(k)} = (-\mathbb{E}[\nabla p_t] + f_t)\mathbf{d}t,$$

with $\operatorname{div} \mathbb{E}[u_t] = 0$, $u_t|_{t=0} = u_0(x)$ and $(\mathcal{L}^T_v u_t)_i := v^j \partial_j u_i + (\partial_i v^j) u^j$.

The Itô formulation is, with $\operatorname{div} u_t = 0$,

$$\mathbf{d}u_t + \mathcal{L}^T_{\mathbb{E}[u_t]} u_t dt + \sum_k \mathcal{L}^T_{\xi^{(k)}} u_t dW_t^{(k)} = \Big(\frac{1}{2}\sum_k \mathcal{L}^T_{\xi^{(k)}}(\mathcal{L}^T_{\xi^{(k)}} u_t) - \mathbb{E}[\nabla p_t] + f_t\Big) dt.$$

Taking the expectation yields a closed equation for $v = \mathbb{E}[u_t]$ given by

$$\partial_t v + \mathcal{L}^T_v v = \frac{1}{2}\sum_k \mathcal{L}^T_{\xi^{(k)}}\big(\mathcal{L}^T_{\xi^{(k)}} v\big) - \mathbb{E}[\nabla p_t] + f_t\,.$$

Thus, $v = \mathbb{E}[u_t]$ obeys the *Lie-Laplacian Navier-Stokes equation (LL NS)*.

Theorem

When LL NS is well-posed, then so is the linear Itô fluctuation equation.

2D LA SALT Euler

The vorticity in 2D LA SALT, understood as a scalar, is governed by the transport law with $\operatorname{div} \mathbb{E}[u_t] = 0 = \operatorname{div} \xi^{(k)}(x)$,

$$\mathbf{d}\omega_t + \mathbb{E}[u_t] \cdot \nabla \omega_t dt + \sum_k \xi^{(k)}(x) \cdot \nabla \omega_t \circ dW_t^{(k)} = 0.$$

This equation implies

$$\int \phi(\omega_t) dx = \int \phi(\omega_0) dx,$$

for any differentiable function ϕ.

In particular, one may choose $\phi(x) = x^p$ and find that all of the L^p-norms of the solution are conserved by the dynamics of 2D LA SALT Euler.

Itô form of 2D LA SALT Euler

In Itô form, 2D LA SALT Euler is given by

$$\mathbf{d}\omega_t + \mathbb{E}[u_t] \cdot \nabla\omega_t dt + \sum_k \xi^{(k)}(x) \cdot \nabla\omega_t dW_t^{(k)} = \frac{1}{2}\sum_k \xi^{(k)}(x) \cdot \nabla\left(\xi^{(k)}(x) \cdot \nabla\omega_t\right) dt.$$

Its expectation is given by

$$\partial_t \mathbb{E}[\omega_t] + \mathbb{E}[u_t] \cdot \nabla \mathbb{E}[\omega_t] = \frac{1}{2}\sum_k \xi^{(k)}(x) \cdot \nabla\left(\xi^{(k)}(x) \cdot \nabla \mathbb{E}[\omega_t]\right) dt.$$

Its fluctuations obey

$$\mathbf{d}\omega_t' + \mathbb{E}[u_t] \cdot \nabla\omega_t' dt + \sum_k \xi^{(k)}(x) \cdot \nabla\omega_t dW_t^{(k)} = \frac{1}{2}\sum_k \xi^{(k)}(x) \cdot \nabla\left(\xi^{(k)}(x) \cdot \nabla\omega_t'\right) dt.$$

Imperial College London

2D LA SALT Euler vorticity variance dynamics

We now want to investigate the dynamics of the variance of the vorticity:

$$\int \mathbb{E}\left[|\omega_t'|^2\right] dxdy = \int \mathbb{E}\left[|\omega_t|^2\right] dxdy - \int \left|\mathbb{E}\left[\omega_t\right]\right|^2 dxdy.$$

The first term on the right is the enstrophy Casimir, which is constant, so

$$\int |\omega_t|^2 dxdy = \int |\omega_0|^2 dxdy \quad \Rightarrow \quad \int \mathbb{E}\left[|\omega_t|^2\right] dxdy = \int \mathbb{E}\left[|\omega_0|^2\right] dxdy.$$

The second term on the right of the variance formula satisfies

$$\frac{1}{2}\frac{d}{dt}\int \left|\mathbb{E}\left[\omega_t\right]\right|^2 dx = -\sum_k \int |\xi^{(k)} \cdot \nabla \mathbb{E}\left[\omega_t\right]|^2 dx,$$

which means that magnitude $|\mathbb{E}\left[\omega_t\right]|$ of the expected vorticity will decay to zero in the absence of forcing, provided that $\{\xi^{(k)}\}_{k\in\mathbb{N}}$ span $\mathbb{R}^3$.

Remark

2D LA SALT dissipates the enstrophy of the mean vorticity by converting it into fluctuations. Thereby, the variance of the vorticity increases under 2D LA SALT dynamics on the initial level set of the enstrophy Casimir.

The nature of stochastic coadjoint motion for LA SALT

Remark

One may regard the expected vorticity equations for 2D LA SALT as a dissipative system embedded into a larger conservative system.

From this viewpoint, the interaction dynamics of the two components of the full LA SALT system dissipates the enstrophy of the mean vorticity by converting it into fluctuations which increase the variance, while preserving the mean total enstrophy.

This dynamics occurs because the total (mean plus fluctuation) vorticity field is being linearly transported along the mean velocity, while the mean vorticity field is decaying in 2D dissipative motion.

This is the nature of stochastic coadjoint motion for LA SALT.
Namely, the Casimirs are preserved by the full LA SALT dynamics, while the equations for the expected dynamics contain dissipative terms.

SALT and LA SALT Burgers equation

Choosing $\ell(u_t) = \frac{1}{2}\int_{S^1} |u_t|^2\, dx$ yields the 1D LA SALT Burgers equation

$$\mathbf{d}u_t + \mathbb{E}[u_t]\,\partial_x u\,\mathbf{d}t + \sum_k \xi^{(k)}\partial_x u_t \circ \mathbf{d}W_t^{(k)} = 0\,,$$

$$\mathbf{d}u_t + \mathbb{E}[u_t]\,\partial_x u\,\mathbf{d}t + \sum_k \xi^{(k)}\partial_x u_t \mathbf{d}W_t^{(k)} = \frac{1}{2}\sum_k \xi^{(k)}\partial_x(\xi^{(k)}\partial_x u_t).$$

Theorem

LA SALT Burgers solutions are regular. (SALT Burgers solutions are not.)

The LA SALT expectation $\mathbb{E}[u_t]$ satisfies a viscous Burgers equation,

$$\partial_t\mathbb{E}[u_t] + \mathbb{E}[u_t]\,\partial_x\mathbb{E}[u_t] = \frac{1}{2}\sum_k \xi^{(k)}\partial_x(\xi^{(k)}\partial_x\mathbb{E}[u_t])\,.$$

This is *regularization by non-locality in probability space.*
Note: the Stratonovich Burgers evolves by transport by push-forward

$$u_t(x) = \phi_{t*}u_0(x), \qquad \mathbb{E}[u_t](x) = \mathbb{E}[\phi_{t*}u_0(x)]\,.$$

Imperial College London

Helicity preservation in SALT & LA SALT Euler equations

The Stratonovich LA SALT Euler fluid motion eqn and its vorticity eqn are

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})(u_t \cdot dx) = -\, dp\,, \quad \text{and} \quad (\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})(\omega_t \cdot dS) = 0\,,$$

where $d(u_t \cdot dx) = \omega_t \cdot dS$ is the vorticity flux (a 2-form), $\omega_t := \mathrm{curl} u_t$ and p is a scalar function. Since $[\mathcal{L}_v, d] = 0$, $d^2 = 0$, and the advection operator $(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})$ obeys the product rule, the motion eqn implies

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})\big((u_t \cdot dx) \wedge (\omega_t \cdot dS)\big) = -\, dp \wedge (\omega_t \cdot dS) = -\, d\big(p\, \omega_t \cdot dS\big)\,,$$

$(u_t \cdot dx) \wedge (\omega_t \cdot dS) = (u_t \cdot \omega_t)\, d^3x$, $\mathcal{L}_{\mathbf{d}X_t}(\Lambda\, d^3x) = \mathrm{div}(\Lambda\, \mathbf{d}X_t)\, d^3x$.

$$\text{Hence, with} \quad \Lambda = u_t \cdot \omega_t\,, \quad \mathbf{d}(\Lambda\, d^3x) = -\, \mathrm{div}\big(\Lambda\, \mathbf{d}X_t + p\, \omega_t\big)\, d^3x\,.$$

Under integration over the spatial domain of the flow,

$$\mathbf{d} \int_{\mathcal{D}} u_t \cdot \mathrm{curl} u_t\, d^3x = -\, \oint_{\partial\mathcal{D}} \big(\Lambda\, \mathbf{d}X_t + p\, \omega_t\big) \cdot dS \;= 0\,,$$

for either vanishing or periodic boundary conditions on $\partial\mathcal{D}$.
Thus, *SALT & LA SALT Euler fluid equations preserve helicity.*

Imperial College London

Helicity preservation in SALT & LA SALT Euler equations

The Stratonovich LA SALT Euler fluid motion eqn and its vorticity eqn are

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})(u_t \cdot dx) = -\,dp\,, \quad \text{and} \quad (\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})(\omega_t \cdot dS) = 0\,,$$

where $d(u_t \cdot dx) = \omega_t \cdot dS$ is the vorticity flux (a 2-form), $\omega_t := \mathrm{curl} u_t$ and p is a scalar function. Since $[\mathcal{L}_v, d] = 0$, $d^2 = 0$, and the advection operator $(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})$ obeys the product rule, the motion eqn implies

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})\big((u_t \cdot dx) \wedge (\omega_t \cdot dS)\big) = -\,dp \wedge (\omega_t \cdot dS) = -\,d\big(p\,\omega_t \cdot dS\big)\,,$$

$(u_t \cdot dx) \wedge (\omega_t \cdot dS) = (u_t \cdot \omega_t)\, d^3x$, $\mathcal{L}_{\mathbf{d}X_t}(\Lambda\, d^3x) = \mathrm{div}(\Lambda\, \mathbf{d}X_t)\, d^3x$.

Hence, with $\quad \Lambda = u_t \cdot \omega_t, \quad \mathbf{d}(\Lambda\, d^3x) = -\,\mathrm{div}\big(\Lambda\, \mathbf{d}X_t + p\,\omega_t\big)\, d^3x\,.$

Under integration over the spatial domain of the flow,

$$\mathbf{d}\int_{\mathcal{D}} u_t \cdot \mathrm{curl} u_t\, d^3x = -\oint_{\partial\mathcal{D}} \big(\Lambda\, \mathbf{d}X_t + p\,\omega_t\big) \cdot dS \;= 0\,,$$

for either vanishing or periodic boundary conditions on $\partial\mathcal{D}$.
Thus, *SALT & LA SALT Euler fluid equations preserve helicity.*

Imperial College London

Helicity preservation in SALT & LA SALT Euler equations

The Stratonovich LA SALT Euler fluid motion eqn and its vorticity eqn are

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})(u_t \cdot dx) = -\,dp\,, \quad \text{and} \quad (\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})(\omega_t \cdot dS) = 0\,,$$

where $d(u_t \cdot dx) = \omega_t \cdot dS$ is the vorticity flux (a 2-form), $\omega_t := \mathrm{curl} u_t$ and p is a scalar function. Since $[\mathcal{L}_v, d] = 0$, $d^2 = 0$, and the advection operator $(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})$ obeys the product rule, the motion eqn implies

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}X_t})\big((u_t \cdot dx) \wedge (\omega_t \cdot dS)\big) = -\,dp \wedge (\omega_t \cdot dS) = -\,d\big(p\,\omega_t \cdot dS\big)\,,$$

$(u_t \cdot dx) \wedge (\omega_t \cdot dS) = (u_t \cdot \omega_t)\, d^3x$, $\mathcal{L}_{\mathbf{d}X_t}(\Lambda\, d^3x) = \mathrm{div}(\Lambda\, \mathbf{d}X_t)\, d^3x$.

$$\text{Hence, with} \quad \Lambda = u_t \cdot \omega_t, \quad \mathbf{d}(\Lambda\, d^3x) = -\,\mathrm{div}\big(\Lambda\, \mathbf{d}X_t + p\,\omega_t\big)\, d^3x\,.$$

Under integration over the spatial domain of the flow,

$$\mathbf{d} \int_{\mathcal{D}} u_t \cdot \mathrm{curl} u_t\, d^3x = -\oint_{\partial\mathcal{D}} \big(\Lambda\, \mathbf{d}X_t + p\,\omega_t\big) \cdot dS \;= 0\,,$$

for either vanishing or periodic boundary conditions on $\partial\mathcal{D}$.
Thus, *SALT & LA SALT Euler fluid equations preserve helicity*.

Example (SALT 2D Euler-Boussinesq (EB) system)

The 2D SALT EB equations are given in Lie–Poisson form by

$$\mathbf{d}F = \{F, h\} = -\int_{\mathbb{T}^2} \begin{bmatrix} \delta F/\delta\mu \\ \delta F/\delta\theta \\ \delta F/\delta D \end{bmatrix}^T \begin{bmatrix} \mathrm{ad}^*_\square \mu & \square \diamond \theta & \square \diamond D \\ \mathcal{L}_\square \theta & 0 & 0 \\ \mathcal{L}_\square D & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta(\mathbf{d}h)/\delta\mu \\ \delta(\mathbf{d}h)/\delta\theta \\ \delta(\mathbf{d}h)/\delta D \end{bmatrix} \mathrm{d}^2x,$$

with $\mathbf{d}x_t := u_t(x_t)\,\mathrm{d}t + \xi(x_t)\circ \mathrm{d}W_t$, $\mu = \mathrm{D}\boldsymbol{u}\cdot \boldsymbol{dx}\otimes \mathrm{d}^2\, x$ and $D = \mathrm{D}\,\mathrm{d}^2\, x$

$$\int_0^t \mathbf{d}h(\mu,\theta,D)\,\mathrm{d}s = \int_0^t \int_{\mathbb{T}^2} \left(\frac{1}{2D}|\mu|^2 + gD\theta y + p(D-1)\right) \mathrm{d}^2x\,\mathrm{d}s$$
$$+ \sum_k \int_0^t \left\langle \mu(x,t)\,,\,\xi_k\right\rangle_{\mathfrak{X}} \circ \mathrm{d}W_s^k\,,$$

$$\frac{\delta(\mathbf{d}h)}{\delta\mu} = \mathbf{d}x_t\,, \quad \frac{\delta(\mathbf{d}h)}{\delta\theta} = gDy\,, \quad \frac{\delta(\mathbf{d}h)}{\delta D} = g\theta y + p - \frac{1}{2}|u|^2$$

Imperial College London

The 2D SALT Euler–Boussinesq (EB) system

The Lie–Poisson form yields the 2D SALT EB system:[2]

$$
\begin{aligned}
\mathbf{d}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\,\mathrm{d}t + \sum_k \boldsymbol{\xi}_k\cdot\nabla\mathbf{u}\circ \mathrm{d}W_t^k + \sum_k u_j\nabla\xi_k^j\circ \mathrm{d}W_t^k \\
= -\nabla(p - |\mathbf{u}|^2/2)\,\mathrm{d}t + g\theta\hat{\boldsymbol{y}}\,\mathrm{d}t, \\
\mathbf{d}\theta + \mathbf{u}\cdot\nabla\theta\,\mathrm{d}t + \sum_k \boldsymbol{\xi_k}\cdot\nabla\theta\circ \mathrm{d}W_t^k = 0, \\
\mathbf{d}\mathrm{D} + \mathrm{div}(\mathrm{D}\mathbf{u}) = 0 \quad \text{and} \quad \mathrm{D} = 1 \Longrightarrow \mathrm{div}\mathbf{u} = 0.
\end{aligned}
$$

[2] Note that for EB the x-y plane is a vertical slice, with $\hat{\boldsymbol{y}}$ in the vertical direction, [AOBdLHT19].

Imperial College London

2D $\boxed{\text{LA}}$ SALT Euler-Boussinesq (EB) system

The 2D LA SALT EB equations are given using the same Lie–Poisson matrix operator as for 2D SALT EB by

$$\mathbf{d}\begin{bmatrix}\mu\\ \theta\\ D\end{bmatrix}=\begin{bmatrix}\mathrm{ad}^*_\square\mu & \square\diamond\theta & \square\diamond D\\ \mathcal{L}_\square\theta & 0 & 0\\ \mathcal{L}_\square D & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbb{E}\left[\delta(\mathbf{d}h)/\delta\mu\right]\\ \mathbb{E}\left[\delta(\mathbf{d}h)/\delta\theta\right]\\ \mathbb{E}\left[\delta(\mathbf{d}h)/\delta D\right]\end{bmatrix},$$

with $\mathbf{d}X_t := \mathbb{E}\left[u_t(x)\right]dt + \xi(x)\circ dW_t$ this becomes

$$\mathbf{d}\begin{bmatrix}\mu\\ \theta\\ D\end{bmatrix}=\begin{bmatrix}\mathrm{ad}^*_\square\mu & \square\diamond\theta & \square\diamond D\\ \mathcal{L}_\square\theta & 0 & 0\\ \mathcal{L}_\square D & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{d}X_t\\ gy\mathbb{E}\left[D\right]\\ gy\mathbb{E}\left[\theta\right]+\mathbb{E}\left[p-\frac{1}{2}|u|^2\right]\end{bmatrix}.$$

The Casimir functionals for this Lie–Poisson matrix operator are given by

$$C_\Phi = \int \mathrm{D}\Phi(\theta)\,d^2x\,,$$

for any differentiable function Φ.

Stratonovich form of LA SALT EB

Since $\mu = \boldsymbol{u} \cdot d\boldsymbol{x} \otimes \mathrm{D}\, \mathrm{d}^2 x$ and $D = \mathrm{D}\, \mathrm{d}^2 x$, we denote $\mu/D = \boldsymbol{u} \cdot d\boldsymbol{x} =: u^\flat$.

The Stratonovich form of LA SALT EB equations may be expressed as

$$
\begin{aligned}
\mathbf{d} u^\flat & + \mathcal{L}_{\mathbb{E}[u]} u^\flat \, \mathrm{d}t + \sum_k \mathcal{L}_{\xi_k} u^\flat \circ \mathrm{d}W_t^k \\
& = - \; d\Big(\mathbb{E}\left[p - \tfrac{1}{2}|u|^2 \right] \Big) \, \mathrm{d}t + g\mathbb{E}\left[\theta\right] \, dy \, \mathrm{d}t \; - gy \, d(\theta - \mathbb{E}\left[\theta\right]) \, \mathrm{d}t \,,
\end{aligned}
$$

$$
\mathbf{d}\theta + \mathcal{L}_{\mathbb{E}[u]} \theta \, \mathrm{d}t + \sum_k \mathcal{L}_{\xi_k} \theta \circ \mathrm{d}W_t^k = 0 \,, \qquad \nabla \cdot \mathbb{E}\left[u\right] = 0.
$$

where $\mathcal{L}_v u^\flat = (v^j \partial_j u_i + (\partial_i v^j) u^j) dx^i =: (\mathcal{L}_v^T u_t)_i dx^i$ and $\mathcal{L}_v \theta = \boldsymbol{v} \cdot \nabla\theta$.

Imperial College London

Itô form of LA SALT EB

Upon passing to the Itô formulation, the LA SALT EB system becomes

$$
\begin{aligned}
\mathbf{d}u^\flat + \mathcal{L}_{\mathbb{E}[u]}u^\flat \,\mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}u^\flat \,\mathrm{d}W_t^k &= -d\Big(\mathbb{E}\Big[p - \tfrac{1}{2}|u|^2\Big]\Big)\,\mathrm{d}t + g\mathbb{E}[\theta]\,dy\,\mathrm{d}t \\
&\quad - gyd(\theta - \mathbb{E}[\theta])\,\mathrm{d}t + \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}u^\flat\,\mathrm{d}t,
\end{aligned}
$$

$$
\mathbf{d}\theta + \mathcal{L}_{\mathbb{E}[u]}\theta\,\mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}\theta\,\mathrm{d}W_t^k = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\theta\,\mathrm{d}t\,, \qquad \nabla\cdot\mathbb{E}[u] = 0.
$$

The composition of Lie derivatives is, for example, $\mathcal{L}_{\xi_k}(\mathcal{L}_{\xi_k}\theta) =: \mathcal{L}^2_{\xi_k}\theta$.
Taking expectation produces the PDE system,

$$
\partial_t\mathbb{E}\left[u^\flat\right] + \mathcal{L}_{\mathbb{E}[u]}\mathbb{E}\left[u^\flat\right] = -\,d\left(\mathbb{E}\left[p - \frac{1}{2}|u|^2\right]\right) + g\mathbb{E}[\theta]\,\hat{\boldsymbol{y}} + \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\mathbb{E}\left[u^\flat\right]
$$

$$
\partial_t\mathbb{E}[\theta] + \mathcal{L}_{\mathbb{E}[u]}\mathbb{E}[\theta] = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\mathbb{E}[\theta]\,.
$$

Imperial College London

Vorticity representation of 2D LA SALT EB

In terms of vorticity $\omega = \hat{\boldsymbol{z}} \cdot \mathrm{curl}\boldsymbol{u}$, the 2D LA SALT EB equations become

$$\mathbf{d}\omega + \mathcal{L}_{\mathbb{E}[u]}\omega \, \mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}\omega \circ \mathrm{d}W_t^k = g(\partial_x\theta)\,\mathrm{d}t\,,$$

$$\mathbf{d}\theta + \mathcal{L}_{\mathbb{E}[u]}\theta \, \mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}\theta \circ \mathrm{d}W_t^k = 0\,.$$

Since the area element $dx \wedge dy$ is constant for 2D incompressible planar flow, the Lie derivatives above of the vorticity $\omega dx \wedge dy$ are given by, e.g., $\mathcal{L}_\xi(\omega dx \wedge dy) = (\xi \cdot \nabla\omega) dx \wedge dy$ and $(\partial_x\theta)dx \wedge dy = d\theta \wedge dy$.

The equations for the expectations (EBX) form a closed PDE sub-system,

$$\partial_t \mathbb{E}\left[\omega\right] + \mathcal{L}_{\mathbb{E}[u]}\mathbb{E}\left[\omega\right] = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\mathbb{E}\left[\omega\right] + g\partial_x\mathbb{E}\left[\theta\right],$$

$$\partial_t \mathbb{E}\left[\theta\right] + \mathcal{L}_{\mathbb{E}[u]}\mathbb{E}\left[\theta\right] = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\mathbb{E}\left[\theta\right].$$

Theorem ([AOBdLHT19])

If EBX eqns are well-posed, so are the linear EB fluctuation transport eqns.

Fluctuation dynamics for 2D LA SALT EB

The fluctuations $\omega' := \omega - \mathbb{E}[\omega]$ and $\theta' := \theta - \mathbb{E}[\theta]$ satisfy the equations:

$$\mathbf{d}\omega' + \mathcal{L}_{\mathbb{E}[u]}\omega'\,\mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}\omega\,\mathrm{d}W_t^k = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\omega'\,\mathrm{d}t + g(\partial_x\theta')\,\mathrm{d}t\,,$$

$$\mathbf{d}\theta' + \mathcal{L}_{\mathbb{E}[u]}\theta'\,\mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}\theta\,\mathrm{d}W_t^k = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}\theta'\,\mathrm{d}t\,.$$

The differential of the second equation leads to the following formula for the $d\theta'$ fluctuation in the first equation, $(\partial_x\theta)dx \wedge dy = d\theta \wedge dy$,

$$\mathbf{d}(d\theta') + \mathcal{L}_{\mathbb{E}[u]}(d\theta')\,\mathrm{d}t + \sum_k \mathcal{L}_{\xi_k}(d\theta)\,\mathrm{d}W_t^k = \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}(d\theta')\,\mathrm{d}t\,,$$

which is needed to complete the analysis of the fluctuation dynamics.

Thus, the system is closed, but the analysis of its variance dynamics is not as straight-forward as for the case of planar vorticity dynamics.

Imperial College London

What is the next step?

Hamilton's principle $\boxed{\surd}$

$+$

Multi-time fast-slow homogenization $\boxed{\surd}$

$\longrightarrow$

Stochastic decomposition $\boxed{\surd}$

$+$

Kunita's Itô-Wenzell (KIW) formula $\boxed{\surd}$

$\longrightarrow$

Eulerian fluid SPDEs with stochastic transport $\boxed{\surd}$

What is the next step? $\boxed{}$

What's next? Do these ideas apply to climate modelling?

"Climate is what you expect. Weather is what you get." [3]

> *There are many questions regarding climate whose answers remain elusive.*
>
> *For example, there is the question of determinism; was it somehow inevitable at some earlier time that the climate now would be as it actually is?*

An *almost intransitive system* is one that can undergo two or more distinct types of behaviour, and will exhibit one type for a long time, but not forever . . . . Can we derive a stochastic almost intransitive system?

[3] Lorenz, E. N., 1995: Climate is what you expect. Unpublished, available at `http://eaps4.mit.edu/research/Lorenz/Climate_expect.pdf`

Lorenz, E. N., 1976: Nondeterministic theories of climatic change. Quaternary Research, 6(4), 495-506.

Daron, J.D. and Stainforth, D.A., 2013. On predicting climate under climate change. Environmental Research Letters, 8(3), p.034021.

One candidate is Lagrangian Averaged (LA) SALT

The LA SALT equations substitute $u_t \to \mathbb{E}[u_t]$ in the Lagrangian path

$$\oint_{C\left(\mathbf{d}x_t = u_t dt + \xi(x)\circ dW_t\right)} u_t \cdot dx \quad\Longrightarrow\quad \oint_{C\left(\mathbf{d}X_t = \mathbb{E}[u_t]dt + \xi(x)\circ dW_t\right)} u_t \cdot dx \,.$$

For example, in the Euler fluid case the modified Kelvin theorem reads,

$$\mathbf{d}\oint_{C(\mathbf{d}X_t)} u_t \cdot dx = \oint_{C(\mathbf{d}X_t)} \left[\mathbf{d}u_t \cdot dx + \mathcal{L}_{\mathbf{d}X_t}(u_t \cdot dx)\right] = 0\,,$$

where $\mathcal{L}_{\mathbf{d}X_t}(u_t \cdot dx)$ denotes the Lie derivative of the 1-form $(u_t \cdot dx)$ with respect to the vector field $\mathbf{d}X_t$ given by

$$\mathbf{d}X_t = \mathbb{E}[u_t]\,dt + \sum_k \xi^{(k)}(x) \circ dW_t\,.$$

The corresponding Euler–Poincaré form of the equations is

$$\mathbf{d}\frac{\delta \ell}{\delta u} + \mathcal{L}_{\mathbf{d}X_t}\frac{\delta \ell}{\delta u} = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\,dt \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}X_t} a = 0\,.$$

Imperial College London

What does LA SALT tell us about extreme events?

When the *expected* Euler–Poincaré equations are written out in $\boxed{\text{Itô form}}$, with $\mu := \frac{\delta \ell}{\delta u}$, we find generalised NS and advected-diffusive equations

$$\frac{\partial}{\partial t}\mathbb{E}[\mu] + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mathbb{E}[\mu] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}[\mu]) = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond \mathbb{E}[a] + \mathbb{E}[\mathbb{F}_\mu] \,,$$

$$\frac{\partial}{\partial t}\mathbb{E}[a] + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mathbb{E}[a] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}[a]) = \mathbb{E}[\mathbb{F}_a] \quad \boxed{\text{Climate}}\,.$$

These $\boxed{\text{Climate}}$ equations predict the expectations $\mathbb{E}[\mu]$ and $\mathbb{E}[a]$ throughout the domain of flow. The Itô $\boxed{\text{Weather}}$ equations for the fluctuations are *linear* drift/stochastic transport relations:

$$\mathbf{d}\mu + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mu + \sum_k \mathcal{L}_{\xi^{(k)}}\mu\, dW_t - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu)\, dt = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\, dt + \mathbb{F}_\mu$$

$$\mathbf{d}a + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}a + \sum_k \mathcal{L}_{\xi^{(k)}}a\, dW_t - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a)\, dt = \mathbb{F}_a \quad \boxed{\text{Weather}}\,.$$

The variance EVOLVES: $\frac{d}{dt}\mathbb{E}\left[\langle |\mu - \mathbb{E}[\mu]|^2\rangle\right] = RHS$

Imperial College London

LA SALT RSW motion

LA SALT RSW motion is governed by the following nondimensional equations for horizontal fluid velocity $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$ with $\mathrm{curl}\mathbf{R}(\mathbf{x}) = 2\Omega(\mathbf{x})\hat{\mathbf{z}}$ and depth D,

$$\mathbf{d}\mathbf{v} - \mathbf{d}\mathbf{X}_t \times \mathrm{curl}\mathbf{v} + \nabla\psi\, dt + \nabla\big(\mathbf{v}\cdot\xi(\mathbf{x})\big)\circ \mathrm{d}W_t = 0\,, \qquad \mathbf{d}D + \nabla\cdot(D\mathbf{d}\mathbf{X}_t) = 0\,,$$

with notation $\psi = (D-B)/(\epsilon\mathcal{F}) + \epsilon|\mathbf{u}|^2/2$, and variable Coriolis parameter $2\Omega(\mathbf{x})$, bottom topography $B = B(\mathbf{x})$, Rossby number ϵ and rotational Froude number $\mathcal{F}$,

$$\epsilon = \frac{\mathcal{U}_0}{f_0 L} \ll 1 \quad \text{and} \quad \mathcal{F} = \frac{f_0^2 L^2}{g B_0} = O(1)\,.$$

The dimensional scales $(B_0, L, \mathcal{U}_0, f_0, g)$ denote equilibrium fluid depth, horizontal length scale, horizontal fluid velocity, reference Coriolis parameter, and gravitational acceleration, respectively.

Imperial College London

(a) The SALT RSW Hamiltonian is obtained from the Legendre transformation in Homework #2 to be

$$\mathbf{d}h(\mu, D) = \int \frac{1}{2\epsilon D}\Big|\mu - D\mathbf{R}(\mathbf{x})\Big|^2 + \frac{(D-B)^2}{2\epsilon\mathcal{F}}dx^1\wedge dx^2\ \mathrm{d}t + \sum_k \int_0^t \Big\langle \mu\,,\, \xi_k(x)\Big\rangle_{\mathfrak{X}} \circ \mathrm{d}W_t^k\,.$$

The variational derivatives are

$$\frac{\delta(\mathbf{d}h)}{\delta\mu} = \mathbf{d}X_t \quad\text{and}\quad \frac{\delta(\mathbf{d}h)}{\delta D} = \frac{(D-B)}{2\epsilon\mathcal{F}} + \epsilon^{-1}|\mathbf{R}|^2 - \frac{|\mu|^2}{2\epsilon D^2}\,.$$

(b) With $\mathbf{d}X_t := \mathbb{E}\left[u_t(x)\right]\mathrm{d}t + \xi(x)\circ\mathrm{d}W_t$ the Lie–Poisson form of the LA SALT RSW equations becomes

$$\mathbf{d}\begin{bmatrix}\mu\\ D\end{bmatrix} = \begin{bmatrix}\mathrm{ad}^*_\Box\mu & \Box\diamond D\\ \mathcal{L}_\Box D & 0\end{bmatrix}\begin{bmatrix}\mathbf{d}X_t\\ \mathbb{E}\left[\dfrac{\delta(\mathbf{d}h)}{\delta D}\right]\end{bmatrix}.$$

Imperial College London

Homework #3.2: What is LA SALT rigid body climate?

(1) The SALT Rigid Body equations may be expressed as

$$\mathbf{d}\Pi = \Pi \times \frac{\partial(\mathbf{d}h)}{\partial \Pi} \quad \text{with} \quad \mathbf{d}h(\Pi) = h(\Pi)\,dt + \Pi \cdot \xi \circ dW_t\,,$$

for a constant $\xi \in so(3) \equiv \mathbb{R}^3$. Discuss the solutions. See arXiv:1601.02249 or `https://doi.org/10.1007/s00332-017-9404-3`.

(2) The LA SALT Rigid Body equations may be expressed as

$$\mathbf{d}\Pi = \Pi \times \mathbb{E}\left[\frac{\partial h}{\partial \Pi}\right] dt + \Pi \times \xi \circ dW_t\,,$$

for a constant $\xi \in so(3) \equiv \mathbb{R}^3$. Discuss the solutions. See [DHL19], arXiv:1908.11481.

What's next? Over to you! Any questions?

[CGH17],[CFH17],[AOBdLHT19],[HT16],[DHL19],[ACH16],[Hol15]

Alexis Arnaudon, Alex L Castro, and Darryl D Holm.
Noise and dissipation on coadjoint orbits.
arXiv preprint arXiv:1601.02249, 2016.

D. Alonso-Oran, A. Bethencourt de Leon, D. D. Holm, and S. Takao.
Modelling the climate and weather of a 2d lagrangian-averaged euler-boussinesq equation with transport noise.
Submitted to J Stat Phys, Special volume on Mathematics of Planet Earth, arXiv:1909.00388, xxx:xx–yy, 2019.

Dan Crisan, Franco Flandoli, and Darryl D Holm.
Solution properties of a 3D stochastic Euler fluid equation.
arXiv preprint arXiv:1704.06989, 2017.

Colin J Cotter, Georg A Gottwald, and Darryl D Holm.
Stochastic partial differential fluid equations as a diffusive limit of deterministic Lagrangian multi-time dynamics.
arXiv preprint arXiv:1706.00287, 2017.

T. D. Drivas, D. D. Holm, and J.-M. Leahy.
Lagrangian averaged stochastic advection by lie transport for fluids.
Submitted to J Stat Phys, Special volume on Mathematics of Planet Earth, arXiv:1908.11481, xxx:xx–yy, 2019.

Darryl D Holm.
Variational principles for stochastic fluid dynamics.
In *Proc. R. Soc. A*, volume 471, page 20140963. The Royal Society, 2015.

Darryl D Holm and Tomasz M Tyranowski.
Stochastic discrete Hamiltonian variational integrators.
arXiv preprint arXiv:1609.00463, 2016.

Imperial College London